Nusselt Numbers Near Entrance of Heat-Exchange Section in Flow Systems

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In the study of heat transfer in thermal-entry regions of hydrodynamically developing or developed viscous flows, the computation of the local Nusselt number as a function of the distance in the direction of flow (z), is of both practical and theoretical importance, deriving from numerous applications with both Newtonian and non-Newtonian fluids, and both laminar and turbulent flows. In view of the wide variety of fluids and flows, the heat transfer problem is of interest for the whole spectrum of z-Peclet numbers, with conduction in the direction of flow ($Pe < 100$), or without ($Pe > 100$) [The stated ranges of Peclet numbers are indeed well established in the literature (Papoutsakis, 1979)].

Our focus in this note is on the behavior of the Nusselt number very near the entrance of the thermal region, namely whether it reaches a finite value or it becomes unbounded as $z \to 0+$, and what determines this behavior. A careful look at Table 1 may justify the scope of the present work. The symbol ∞ in Table 1 is used to denote unboundedness for the Nusselt number; the question mark implies ambiguity regarding the stated limit. Finally, Table 1 is by no means complete, but merely represents the existing information in the literature.

As it may be seen from Table 1, there exists not only ambiguity about the behavior of the Nusselt number as $z \to 0+$, but also a number of contradictory reports. It is also certain that the factors which determine the (true) behavior of the Nusselt number are very little understood. A number of reasons could have been stated for the existing ambiguity and contradictions, but the essential reasons appear to be lack of analytical results, that would permit a careful study, and the convergence difficulties that all numerical techniques face near boundary discontinuities.

This study becomes possible by recently obtained analytical results for the extended Graetz problem (Papoutsakis, 1979; Papoutsakis et al., 1980a, 1980b) with fully developed flows. We will discuss later that the conclusions of this study hold most probably true for the case of developing flows as well.

ANALYSIS

Consider fully developed flow in a tube, with some specified velocity profile, and specified wall boundary conditions for $z < 0$ (upstream section) and $z > 0$ (thermal development section). Assuming constant physical properties, negligible viscous dissipation and axisymmetry, the dimensionless energy equation for any combination of wall boundary conditions, may be written as (Papoutsakis, 1979; Papoutsakis et al., 1980a; 1980b):

$$-\frac{1}{Pe^2}\frac{\partial^2\Theta}{\partial\xi^2} - \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Theta}{\partial\eta}\right) + v(\eta)\frac{\partial\Theta}{\partial\xi} = 0$$

$$-\infty < \xi < \infty \quad 0 < \eta < 1 \qquad (1)$$

It is of course understood that the dimensionless temperature is defined differently depending on the employed wall boundary conditions. The corresponding definition of the local Nusselt number is given by,

$$Nu = \frac{-2\left(\frac{\partial\Theta}{\partial\eta}\right)_{\eta=1}}{\Theta_b - \Theta_w} \qquad (2)$$

where Θ_w is the dimensionless wall temperature and Θ_b the dimensionless bulk temperature defined in the notation section.

The terminology for the wall boundary conditions is explained in the footnote of Table 1. For every problem we have a combination of wall boundary conditions, e.g. Neumann-Dirichlet, the first one referring to the upstream section ($z < 0$) and the second one to the heat-exchange section ($z > 0$) of the tube. Now we study the problem for the various possible combinations of wall boundary conditions.

I. Dirichlet-Dirichlet. This is the case of a step change in the wall temperature, namely,

$$\Theta = \begin{cases} 1 & \xi < 0 \\ 0 & \xi > 0 \end{cases}, \quad \eta = 1 \qquad (3)$$

We seek to determine the limit of $Nu(\xi)$ for $\xi \to 0$, utilizing the solution of this extended Graetz problem which has been recently derived (Papoutsakis et al., 1980a) and has as follows:

$$\Theta(\xi,\eta) = -2\sum_{j=1}^{\infty}\frac{\phi_{j2}(1)\phi_{j1}(\eta)}{\lambda_j\|\underset{\sim}{\phi}_j\|^2} + \sum_{j=1}^{\infty}\frac{2\phi_{j2}^{+}(1)e^{\lambda_j^{+}\xi}}{\lambda_j^{+}\|\underset{\sim}{\phi}_j^{+}\|^2}\phi_{j1}^{+}(\eta) \quad \xi \leq 0 \qquad (4)$$

$$\Theta(\xi,\eta) = \sum_{j=1}^{\infty}\frac{-2\phi_{j2}^{-}(1)e^{\lambda_j^{-}\xi}}{\lambda_j^{-}\|\underset{\sim}{\phi}_j^{-}\|^2}\phi_{j1}^{-}(\eta) \quad \xi \geq 0 \qquad (5)$$

Note that the solution utilizes both the positive and negative eigenvalues of the eigenvalue problem:

$$\underset{\sim}{L}\,\underset{\sim}{\phi}_j = \lambda_j\,\underset{\sim}{\phi}_j \qquad (6a)$$

$$\phi_{j1}(1) = \phi_{j2}(0) = 0 \qquad (6b)$$

where

$$\underset{\sim}{\phi}_j(\eta) = \begin{bmatrix}\phi_{j1}(\eta)\\ \phi_{j2}(\eta)\end{bmatrix}$$

while from Eq. 6a we obtain,

$$Pe^2 v(\eta)\phi_{j1}(\eta) - \frac{Pe^2}{2\eta}\phi_{j2}'(\eta) = \lambda_j\,\phi_{j1}(\eta) \qquad (7)$$

$$2\eta\phi_{j1}'(\eta) = \lambda_j\phi_{j2}(\eta) \qquad (8)$$

The linear, matrix, differential operator $\underset{\sim}{L}$ is defined in (Papoutsakis et al., 1980a), where it is also shown that the first term of the righthand side of Eq. 4 is equal to 1. In fact, this term is only conditionally convergent [which is due to the fact that it is quite

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a "strain" on a $\phi_{j1}(\eta)$ series in view of Eq. 6b to give us a function which has a unit value at $\eta = 1$] and thus it cannot be differentiated in the classical sense. It may be readily seen from Eqs. 4 and 5 that $\Theta(\xi, \eta)$ is a continuous function at $\xi = 0$ except at $\eta = 1$; thus, Θ_b is a continuous function of ξ everywhere in view of its definition. Note also that Θ_b is always different from Θ_w except for $\xi \rightarrow -\infty$. Θ_w is indeed given by Eq. 3 and hence it is the negative Heaviside (step) function of ξ. Nevertheless, the denominator of Eq. 2 is a well-defined, discontinuous (at $\xi = 0$), *finite*, function of ξ. In fact, the value that we may assign to Θ_w at $\xi = 0$ does not affect the $Nu(\xi)$ at $\xi = 0$, since as we shall see soon, the numerator of Eq. 2 becomes unbounded as $\xi \rightarrow 0$. Therefore, we concentrate on the numerator of Eq. 2. The η derivative of Θ at $\xi = 0$ may be obtained from either Eq. 4 or 5, in view of the continuity of $\Theta(\xi, \eta)$ at $\xi = 0$, however, this can be done only in the context of the generalized functions (for example, Vladimirov, 1971) since, as we saw, the first term of the righthand side of Eq. 4 cannot be differentiated at $\eta = 1$ in the classical sense. Thus, from Eq. 4 and using Eq. 8 we obtain,

$$\frac{\partial\Theta}{\partial\eta}(0,\eta) = -\sum_{j=1}^{\infty}\frac{\phi_{j2}(1)\phi_{j2}(\eta)}{\|\phi_j\|^2\,\eta} + \sum_{j=1}^{\infty}\frac{\phi_{j2}^{+}(1)\phi_{j2}^{+}(\eta)}{\|\phi_j^{+}\|^2\,\eta} = -\delta(\eta-1) + \left\{\frac{\partial\Theta}{\partial\eta}(0,\eta)\right\} \quad (9)$$

Following the procedure we used in Appendix III of our earlier work (Papoutsakis et al., 1980a) for a similar result, it can be readily shown that indeed,

$$\delta(\eta - 1) = \sum_{j=1}^{\infty}\frac{\phi_{j2}(1)\phi_{j2}(\eta)}{\|\phi_j\|^2\eta} \quad (10)$$

In Eq. 9, $\{\partial\Theta/\partial\eta(0,\eta)\}$ denotes the classical derivative, if it exists, or zero if it doesn't exist (Vladimirov, 1971, p. 73). Thus,

$$\left\{\frac{\partial\Theta}{\partial\eta}(0,\eta)\right\} = \begin{cases} 0 & \eta = 1 \quad (11a) \\ \displaystyle\sum_{j=1}^{\infty}\frac{\phi_{j2}^{+}(1)\phi_{j2}^{+}(\eta)}{\|\phi_j^{+}\|^2\,\eta} & 0 < \eta < 1 \quad (11b) \end{cases}$$

Since $\partial\Theta/\partial\eta\,(\xi,1)$ is positive for $\xi < 0$ and negative for $\xi > 0$, Eq. 9 may be interpreted to imply (Vladimirov, 1971, pp. 63-75) in classical terms that,

$$\lim_{\xi\rightarrow 0-}\frac{\partial\Theta}{\partial\eta}(\xi,1) = -\lim_{\xi\rightarrow 0+}\frac{\partial\Theta}{\partial\eta}(\xi,1) = \infty \quad (12)$$

from which, and by virtue of Eqs. 2 and 3, we obtain,

$$\lim_{\xi\rightarrow 0-} Nu(\xi) = \lim_{\xi\rightarrow 0+} Nu(\xi) = \infty \quad (13)$$

Note that Eq. 13 is true for the whole spectrum of the axial Peclet numbers. For the case of $Pe \rightarrow \infty$, Eq. 13 can be readily derived from first principles. Indeed, for $Pe = \infty$

$$\Theta(0,\eta) = \begin{cases} 1 & 0 < \eta < 1 \\ 0 & \eta = 1 \end{cases} \quad (14)$$

thus,

$$\frac{\partial\Theta}{\partial\eta}(0,\eta)|_{\eta=1} = -\delta(\eta - 1) \quad (15)$$

Hence,

$$\frac{\partial\Theta}{\partial\eta}(0,\eta) = 0 - \delta(\eta - 1) = \left\{\frac{\partial\Theta}{\partial\eta}(0,\eta)\right\} - \delta(\eta - 1) \quad (16)$$

from which Eq. 13 is obtained as before. For the other limiting case, $Pe \rightarrow 0$, Eq. 12 (and thus, Eq. 13) is a rather well-known result (Morse and Feshbach, 1953, pp. 1176-1178).

II. Neumann-Dirichlet or Neumann-Robin. These two cases are handled together since their solutions have the same mathematical form and properties and lead to qualitatively similar results. Physically, the upstream section is now thermally insulated while on the wall of the heating section the temperature or a linear combination of the temperature and its derivative are prescribed. These are mixed boundary value problems and their solution may be obtained in the form (Papoutsakis, 1979, Chapter 6),

$$\Theta(\xi,\eta) = \sum_{j=1}^{\infty} A_j^{-}\, e^{\lambda_j^{-}\xi}\,\phi_{j1}^{-}(\eta) \qquad \xi \geq 0 \quad (17)$$

$$\Theta(\xi,\eta) = \sum_{j=1}^{\infty}\frac{\psi_{j1}(1)\psi_{j1}(\eta)}{k_j\|\psi_j\|^2} + \sum_{j=1}^{\infty}\frac{\psi_{j1}^{+}(1)\psi_{j1}^{+}(\eta)}{\|\psi_j^{+}\|^2}\left(\frac{1}{k_j^{+}} + 2X_j\right)e^{k_j^{+}\xi} \qquad \xi \leq 0 \quad (18)$$

where the set $\{\lambda_j, \underset{\sim}{\phi}_j(\eta)\}$ satisfies Eq. 6 for the Neumann-Dirichlet problem and Eq. 6a with

$$\phi_{j2}(0) = \phi_{j1}'(1) + B\phi_{j1}(1) = 0 \quad (19)$$

for the Neumann-Robin problem, while $\{k_j, \underset{\sim}{\psi}_j\}$ satisfies:

$$\underset{\sim}{L}\,\underset{\sim}{\psi}_j = k_j\,\underset{\sim}{\psi}_j \quad (20a)$$

$$\psi_{j2}(0) = \psi_{j2}(1) = 0 \quad (20b)$$

Constants A_j^{-} and X_j are computed as shown previously (Papoutsakis, 1979). Furthermore, we may readily show that (except for the Neumann-Dirichlet problem with $Pe = \infty$) the so-

TABLE 1. NUSSELT NUMBERS NEAR ENTRANCE OF HEAT-EXCHANGE SECTION.

Reference	Axial Domain (z)	Wall Boundary Conditions $z < 0$	Wall Boundary Conditions $z > 0$	$\lim Nu(z)$ $z \rightarrow 0+$	Remarks
Agrawal (1960)	$(-\infty,\infty)$	Dirichlet	Dirichlet	∞	ES
Schneider (1956)	$(-\infty,\infty)$	Dirichlet	Dirichlet	∞	IFG
Hennecke (1968)	$(-\infty,\infty)$	Dirichlet	Dirichlet	∞	ES
Schmidt and Zeldin (1970)	$[0,\infty)$	—	Dirichlet	∞?	IFG
Tan and Hsu (1970)	$[0,\infty)$	—	Dirichlet	∞	IFG
Michelsen and Villadsen (1974)	$(-\infty,\infty)$	Neumann	Dirichlet	∞	ES
Tan and Hsu (1972)	$(-\infty,\infty)$	Neumann	Dirichlet	finite	IFG
Schneider (1956)	$(-\infty,\infty)$	Robin	Robin	finite?	IFG
Hsu (1968)	$[0,\infty)$	—	Robin	∞?	IFG
Dang (1978)*	$(-\infty,\infty)$	Neumann	Neumann	finite	IFG
Hennecke (1968)	$(-\infty,\infty)$	Neumann	Neumann	finite	IFG
Hennecke (1968)	$(-\infty,\infty)$	Neumann	Neumann	∞, for $Pe \rightarrow \infty$	IFG
Hsu (1967)	$[0,\infty)$	—	Neumann	∞?	IFG
Pirkle and Sigillito (1972)	$[0,\infty)$	—	Neumann	∞	IFG

* With chemical reaction in $[0, \infty)$. No information for $Pe \rightarrow \infty$.
(?), not certain; ES, explicitly stated; IFG, inferred from graph.
Dirichlet, prescribed temperature; *Neumann*, prescribed heat flux; *Robin*, a prescribed linear combination of temperature and heat flux.

lution (Eqs. 17 and 18) is continuous and unconditionally convergent at $\xi = 0$ and for all $\eta \epsilon [0,1]$. Thus, $\partial\Theta/\partial\eta\ (0,1)$ may be readily computed, by direct differentiation of either Eq. 17 or 18, to be of finite value. Incidentally, the first series of the right hand side of Eq. 18 may be shown to be equal to 1 for all $\eta\epsilon[0,1]$ (Papoutsakis, 1979). Again, since $\Theta_b \neq \Theta_w$ and since now both Θ_b and Θ_w are continuous at $\xi = 0$ (Papoutsakis, 1979), $Nu(\xi)$, by virtue of Eq. 2, is *continuous*, and *finite* at $\xi = 0$. For $Pe = \infty$, the Neumann-Dirichlet is the same as the Dirichlet-Dirichlet problem, Eq. 17 is the same as Eq. 5; thus, Eq. 13 is true. For the Neumann-Robin problem with $Pe = \infty$, $\Theta(0, \eta) = 1$ for all $\eta\epsilon[0,1]$; thus, $\Theta_b = \Theta_w$ at $\xi = 0$, while $\Theta_w > \Theta_b$ for $\xi > 0$; hence, $\lim_{\xi\to 0+} Nu(\xi) = \infty$.

III. Robin-Robin. This case may analyzed in a way similar to that of Case II, and again $Nu(\xi)$ is found to be continuous and finite at $\xi = 0$ for all Peclet numbers.

IV. Neumann-Neumann. Physically, the upstream section is thermally insulated while in the heat-exchange section the fluid is heated with a prescribed, finite, heat flux, i.e.,

$$\frac{\partial\Theta}{\partial\eta}(\xi,1) = \begin{cases} 1 & \xi \geq 0 \quad (21a) \\ 0 & \xi < 0 \quad (21b) \end{cases}$$

It can be shown (Papoutsakis et al., 1980b) that $\Theta(\xi,\eta)$ is continuous at $\xi = 0$ for all $\eta\epsilon[0,1]$ and thus both Θ_b and Θ_w are also continuous and different (for $Pe \neq \infty$) at $\xi = 0$. Hence, by virtue of Eqs. 2 and 21, $Nu(\xi)$ is *discontinuous* but *finite* at $\xi = 0$. The same result may be similarly shown for the problem studied by Dang (1978) with $Pe \neq \infty$. For $Pe = \infty$,

$$\Theta(0,\eta) = 1 \quad \text{all} \quad \eta\epsilon[0,1] \qquad (22)$$

Hence, $\Theta_b = \Theta_w$ at $\xi = 0$, and thus by virtue of Eqs. 21a, 2 and $\Theta_w > \Theta_b$ for $\xi > 0$, $\lim_{\xi\to 0+} Nu(\xi) = \infty$.

Naturally, all the above conclusions are true for any other, usual, problem geometry.

DISCUSSION

The analysis above has shown that the Nusselt number near the entrance of the heating section and for $Pe \neq \infty$ is determined by the boundary discontinuity at $\xi = 0$. A temperature discontinuity on the wall at $\xi = 0$ results in an unbounded radial derivative of the temperature and thus by definition (Eq. 2) the Nusselt number becomes unbounded. If the temperature on the wall is continuous at $\xi = 0$, the numerator of Eq. 2 is also continuous and finite at $\xi = 0$ and so is the Nusselt number provided that the denominator of Eq. 2 does not become zero at $\xi = 0$. A zero denominator for Eq. 2 is resulting from $Pe = \infty$ with certain combinations of wall boundary conditions. In no case however does the numerator of Eq. 2 become unbounded with the denominator becoming simultaneously zero at $\xi = 0$. Since the Nusselt number behavior around $\xi = 0$ is determined by the wall discontinuities at $\xi = 0$, it is only natural to expect that the presently derived conclusions will hold qualitatively true for problems with hydrodynamically developing flows as well. This indeed has been found to be the case for a Neumann-Dirichlet problem with a hydrodynamically developing flow (Christiansen and Kelsey, 1972).

All past studies (Table 1) employing a semi-infinite axial domain with a finite Peclet number have utilized the boundary condition of Eq. 22, which however is inconsistent with the inclusion in the analysis of axial conduction (Papoutsakis et al., 1980a, 1980b). Yet, the boundary condition of Eq. 22 results in an erroneously unbounded Nusselt number at $\xi = 0$ as we have presently shown. Other unclear results in Table 1 may be attributed to the solution techniques utilized.

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NOTATION

B = Biot number
k_j, k_j^+ = eigenvalues, Eq. (20)
$\underset{\sim}{L}$ = a linear, matrix, differential operator
Nu = Nusselt number
Pe = Peclet number in the direction of flow
T = fluid temperature
T_1, T_D = characteristic temperatures that depend upon the particular problem (Papoutsakis, 1979; Papoutsakis et al., 1980a; 1980b)
v = dimensionless velocity profile
z = axial variable
$\delta(\ )$ = delta function
ϵ = belongs to
ξ = dimensionless z
η = dimensionless radial variable
Θ = dimensionless temperature ($\equiv (T - T_1)/T_D$)
Θ_b = bulk Θ ($\equiv 4 \int_0^1 \Theta v(\eta)\eta d\eta$)
Θ_w = wall Θ
λ_j, λ_j^+, λ_j^- = eigenvalues, positive and negative eigenvalues, Eq. 6
ϕ_j, $\phi_j^\pm$ = eigenvectors, Eq. 6
$\underset{\sim}{\psi}_j^\pm$, $\psi_{j1}^\pm$, $\psi_{j2}^\pm$ = eigenvectors and their first and second components, Eq. 20

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